

On convergence of a semi-analytical method for American option pricing [☆]

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Received 22 December 2003

Available online 19 October 2005

Submitted by William F. Ames

Abstract

We examine the valuation of American put options by a semi-analytical method, and obtain the prior estimate and the convergence of the approximate solution. Our proofs are based on the embedding theorem in Sobolev space and the theory of functional analysis, in particular, the theory of weak compactness. The results in this paper theoretically confirm empirical observations that these methods are accurate and computationally efficient.

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Keywords: American option; Free boundary; Prior estimate; Semi-analytic method; Convergence

1. Introduction

Based on a “local no arbitrage” assumption, Black and Scholes [4] and Merton [17] developed the theory that is well known as the Black–Scholes formula, which can be applied to derive a

[☆] This research is partially supported by a CERG grant of Hong Kong RGC (Grant CityU1156/04E), a SRG grant of City University of Hong Kong (Grant 7001545), and a project of National Natural Science Foundation of China (Grant 70003002).

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closed form solution for European option pricing (see Duffie [9]). The Black–Scholes model has since been widely applied to value options and other derivatives (for example, see Kwok [14], Schwartz [21] and Willmott [24]). A similar solution can be obtained for American calls on non-dividend-paying stocks since they are not rationally exercised early. On the other hand, American puts cannot be valued in closed form because they may be exercised early for optimal profit making (Geske and Johnson [10], Huang et al. [12], MacMillan [16], Parkinson [20], Tilly [22] and Villeneuve [23]). Unfortunately, the vast majority of listed options are of the American style and are thus subject to early exercises. This poses a major challenge in mathematical finance to find an efficient solution for the American option pricing problem.

Mathematically, the non-availability of an analytic solution for American options results from the difficulty in obtaining a simple expression for the optimal exercise boundary. The optimal exercise boundary of an American option is an unknown function to be determined together with the solution to the valuation problem. Here an exercise boundary consists of a time path of critical stock prices at which early exercises occur. This difficulty has attracted efforts trying in different directions to resolve it, including analytic approximation, numerical computation, and the so-called semi-analytic methods. Naturally, an immediate question for each of these approaches is whether it converges to the exact solution (for which a closed form is not available).

Though empirical and simulation results often show good supporting evidence, convergence has been difficult to theoretically prove for numerical methods and semi-analytic methods. Based on special properties of put and call options, Jaillet et al. [13] have been able to provide a complete justification of Brennan–Schwartz algorithm, a finite element approach (Brennan and Schwartz [5]), for valuation of American put options.

In this work, we follow the analytical method of lines of Carr and Faguet [8], and prove the convergence of the approximate solution. Though our results are obtained for the analytical method of lines, convergence can also be proved for the randomization method of Carr [7]. Moreover, the results in this paper are still valid for the approximate solution of other American options such as those in Meyer [18], Meyer and Van der Hoek [19].

The analytic method of lines of Carr and Faguet is a semi-analytic method that combines analytic approximation and numerical computation. Analytic approximation methods usually bypass the difficulty of directly solving the general problem by restricting the solution space in exchange of a closed form solution. Numerical methods, on the other hand, usually rely on discretizing both time and space and then solving the resulting finite problem numerically. The analytical methods of lines by Carr and Faguet [8] discretizes the problem along the time axis and transforms the problem into an ordinary differential equation problem with a free boundary (at each fixed time value). The solution is obtained by solving each time level in the closed form and determining the coefficients through boundary conditions.

Consider the price of a put option $P(S, t)$ that satisfies the Black–Scholes partial differentiable equation (PDE)

$$\frac{1}{2}\sigma^2 S^2 P_{SS}(S, t) + rSP_S(S, t) - rP(S, t) + P_t(S, t) = 0, \quad t \in (0, T), \quad (1.1)$$

where $r > 0$ is the constant riskless interest rate, $\sigma > 0$ is the constant volatility rate, S is the underlying asset price, T is the given expiration date, and t is the time variable. This is a backward partial differential equation. We may take a transformation (time to maturity) $\tau = T - t$ to change Eq. (1.1) into the usual forward second-order partial differential equation

$$\frac{1}{2}\sigma^2 S^2 P_{SS}(S, \tau) + rSP_S(S, \tau) - rP(S, \tau) = P_\tau(S, \tau), \quad \tau \in (0, T). \quad (1.2)$$

The price $P(S, \tau)$ of the American put option also satisfies the following initial-boundary value conditions:

$$\begin{cases} P(S, 0) = (X - S)^+, \\ P(\underline{S}(\tau), \tau) = X - \underline{S}(\tau), \\ P_S(\underline{S}(\tau), \tau) = -1, \\ P(\infty, \tau) = 0, \end{cases} \quad (1.3)$$

where $X > 0$ is the constant strike price, $\underline{S}(\tau)$ is the unknown exercise boundary. For this reason the problem (1.2)–(1.3) is usually referred to as a free boundary problem.

Equation (1.2) on its own is a simple linear partial differential equation. However, since the initial-boundary value conditions (1.3) are not linear, the problem (1.2) with initial-boundary conditions (1.3) is non-linear. Its solution cannot have a closed form similar to the solution for the problem of European option pricing. The intrinsic difficulty for the valuation of American options lies on the unknown free boundaries associated with the early exercise feature, which renders the parabolic problem non-linear. Therefore, it is widely acknowledged that an analytical formula does not exist for the value of an American option where early exercise may be optimal.

In the analytic method of lines of Carr and Faguet [8], the time variable in Eq. (1.2) is discretized. At each fixed time level, the problem (1.2)–(1.3) is changed into a free boundary problem of an ordinary differential equation on the underlying asset price variable S . Then the closed form option pricing formula can be obtained for each of them. The coefficients in these closed form solutions can in turn be determined by continuity of these functions at the time boundary, starting from the first level. It has been observed that numerical results indicate that the approximation is both accurate and computationally efficient. However, solutions obtained from the two methods are still approximate solutions and not exact solutions for the problem (1.2)–(1.3). Then a natural question is that: to what extent does this kind of semi-analytical solution approach approximately the exact solution to the problem (1.2)–(1.3)? Can it converge to the exact solution? This question is not only important in theory, but also is valuable in practice. However, there are some difficulties to rigorously prove the convergence of the approximate solution to the American put option price. For example, it is known that the unknown boundary $\underline{S}(\tau)$ for an American option has the following asymptotic behavior (Barleo et al. [2]):

$$\underline{S}(\tau) \sim X \left(1 - \sigma \sqrt{\tau \log \frac{1}{\tau}} \right), \quad \text{when } \tau \rightarrow 0.$$

So $\underline{S}'(\tau)$, $\theta = \frac{\partial P(S, \tau)}{\partial \tau}$ and $\Gamma = \frac{\partial^2 P(S, \tau)}{\partial S^2}$ have a singularity when $S \rightarrow X$ and $\tau \rightarrow 0$. But the convergence of the approximate solution strongly depends on θ and Γ of the exact solution for the American put option. To overcome this difficulty of the singularity, our convergence result is based on a concept of weak convergence in the study of functional analysis (see, e.g., Adams [1] and Loins [15]). Using this weak convergence result, we are able to derive the strong convergence result through a mathematical result in Sobolev space.

In Section 2 and Appendices A and B we present the analytical method of lines and some mathematical preliminaries for our proof. Section 3 shows a prior estimate on the approximate solution provided by the analytical method of lines. Section 4 proves the convergence of the approximate solution. Section 5 concludes the paper with remarks.

2. Analytical method of lines

The main line of thought in the analytical method of lines is that, when the time differential $\frac{\partial P(S, \tau)}{\partial \tau}$ is replaced by the time difference $\frac{P(S, n\Delta\tau) - P(S, (n-1)\Delta\tau)}{\Delta\tau}$, the problem (1.2)–(1.3) is transformed into a free boundary problem of an ordinary differential equation for the variable S at some time layer ($\tau = n\Delta\tau$). This is a linear ordinary differential equation with the free boundary value, thus the solution to the ordinary differential equation at every layer can be represented by a closed form. Because of this feature, this type of method is called a semi-analytical method.

Let $\Delta\tau = \frac{T}{N+1}$ ($N = 1, 2, \dots$) be the length of difference step in the direction τ . Then $P^n(S)$ approximates the solution $P(S, \tau)$ of the problem (1.2)–(1.3) at the time $\tau = n\Delta\tau$, and \underline{S}^n approximates the boundary value \underline{S} at the time $\tau = n\Delta\tau$. Thus the approximate equation of the corresponding equation (1.2) is

$$\frac{P^n(S) - P^{n-1}(S)}{\Delta\tau} = \frac{1}{2}\sigma^2 S^2 P_{SS}^n(S) + rSP_S^n(S) - rP^n(S),$$

$$n = 1, \dots, N, \quad S \in (\underline{S}^n, \infty), \quad (2.1)$$

and the approximate relation of the corresponding initial-boundary value conditions (1.3) is

$$\begin{cases} P^0(S) = (X - S)^+, & S \in (\underline{S}^n, \infty), \\ P^n(\underline{S}^n) = X - \underline{S}^n, \\ P_S^n(\underline{S}^n) = -1, \\ P^n(\infty) = 0. \end{cases} \quad (2.2)$$

We can solve the problem (2.1)–(2.2) “layerwise” from $n = 1$ to $n = N$, then obtain the representation of $(P^n(S), \underline{S}^n)$ given in Appendix A.

For simplicity of calculation, we take the transformation as follows:

$$\begin{cases} S = Xe^y \quad \text{or} \quad y = \log \frac{S}{X}, & S > 0, \\ \underline{y}(\tau) = \log \frac{\underline{S}(\tau)}{X}, & \underline{y}^n = \log \frac{\underline{S}^n}{X}, \\ \bar{P}(y, \tau) = \frac{1}{X} P(Xe^y, \tau), & \bar{P}^n(y) = \frac{1}{X} P^n(Xe^y). \end{cases} \quad (2.3)$$

We also write $\bar{P}(y, \tau)$ and $\bar{P}^n(y)$ as $P(y, \tau)$ and $P^n(y)$, respectively. Thus the problems (1.2)–(1.3) and (2.1)–(2.2) are transformed into the following form:

$$P_\tau(y, \tau) = \mathcal{L}P(y, \tau), \quad y \in (\underline{y}(\tau), \infty), \quad \tau \in (0, T), \quad (2.4)$$

$$\begin{cases} P(y, 0) = (1 - e^y)^+, & y \in (\underline{y}(0), \infty), \\ P(\underline{y}(\tau), \tau) = 1 - e^{\underline{y}(\tau)}, & \tau \in (0, T), \\ P_y(\underline{y}(\tau), \tau) = -e^{\underline{y}(\tau)}, & \tau \in (0, T), \\ P(\infty, \tau) = 0, & \tau \in (0, T), \end{cases} \quad (2.5)$$

and

$$P^n(y) - \Delta\tau \mathcal{L}P^n(y) = P^{n-1}(y), \quad y \in (\underline{y}^n, \infty), \quad (2.6)$$

$$\begin{cases} P^0(y) = (1 - e^y)^+, & y \in (\underline{y}^n, \infty), \\ P^n(\underline{y}^n) = 1 - e^{\underline{y}^n}, \\ P_y^n(\underline{y}^n) = -e^{\underline{y}^n}, \\ P^n(\infty) = 0, \end{cases} \quad (2.7)$$

where the operator \mathcal{L} in the relations (2.4) and (2.6) is

$$\mathcal{L} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial}{\partial y} - r, \quad (2.8)$$

which is a constant coefficients second-order partial differential operator.

In addition, we define

$$P^n(y) = 1 - e^y, \quad y \in (\underline{y}(\infty), \underline{y}^n), \quad (2.9)$$

thus $P^n(y) \in H^1(\underline{y}(\infty), \infty)$ (the definition for the function space $H^1(\underline{y}(\infty), \infty)$ is in Appendix B).

From Carr and Faguet [8], we then have

$$\underline{y}(\infty) = \log \frac{S(\infty)}{X} = -\log \left(1 + \frac{\sigma^2}{2r}\right). \quad (2.10)$$

3. Prior estimate

In order to obtain the convergence of the approximate solution sequences $\{P^n(y)\}$ and $\{\underline{y}^n\}$ of the problem (2.6)–(2.7), we consider the prior estimate of them.¹ Our ideas are as follows. From the problem (2.6)–(2.7) and the definition (2.9), we establish a uniform upper bound independent of n in the function space $H^1(\underline{y}(\infty), \infty)$ of $P^n(y) \in H^1(\underline{y}(\infty), \infty)$, that is, $\|P^n(\cdot)\|_{H^1(\underline{y}(\infty), \infty)} \leq C$. In the next section, we obtain some subsequence of $\{P^n(y)\}$ (which we still denote as $\{P^n(y)\}$) which weakly converges to some function $\tilde{P}(y, \tau)$ in the space $H^1(\underline{y}(\infty), \infty)$ from the theory in functional analysis. According to the compactness of the Sobolev embedding theorem, we can choose some subsequence of $\{P^n(y)\}$ (which we still denote as $\{P^n(y)\}$) again such that $P^n(y)$ uniformly converges to $\tilde{P}(y, \tau)$ in the space $C(\underline{y}(\infty), \infty)$, that is,

$$\lim_{n \rightarrow \infty} \|P^n(y) - \tilde{P}(y, \tau)\|_{C(\underline{y}(\infty), \infty)} = 0.$$

Finally, we show that the limit function $\tilde{P}(y, \tau)$ is the generalized solution to the problem (2.4)–(2.5) (the definition of the generalized solution is in Appendix B).

Now we make the estimate for $\|P^n(y)\|_{L^2(\underline{y}(\infty), \infty)}$. Multiplying the two sides of (2.6) by $P^n(y)$, then integrating on the interval $(\underline{y}^n, \infty)$, we have

$$\int_{\underline{y}^n}^{\infty} [P^n(y) - \Delta \tau \mathcal{L} P^n(y)] P^n(y) dy = \int_{\underline{y}^n}^{\infty} P^{n-1}(y) P^n(y) dy. \quad (3.1)$$

First, we consider the left-hand side of the relation (3.1),

¹ From theoretical methodology, we should be able to obtain the estimate from the representation $\{P^n(y)\}$ directly from (2.4) and (2.5). But that would be a tedious and difficult task. Instead, we obtain a concise estimate from the problem (2.6)–(2.7).

$$\begin{aligned} \int_{\underline{y}^n}^{\infty} \mathcal{L} P^n(y) P^n(y) dy &= -\frac{1}{2} \sigma^2 \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 - r \|P^n\|_{L^2(\underline{y}^n, \infty)}^2 \\ &\quad + \frac{1}{2} \sigma^2 [P_y^n(y) P^n(y)]|_{\underline{y}^n}^{\infty} + \frac{1}{2} \left(r - \frac{1}{2} \sigma^2\right) [P^n(y)]^2|_{\underline{y}^n}^{\infty}. \end{aligned}$$

Substituting it into the relation (3.1) and using the Cauchy inequality, we then have

$$\begin{aligned} &\|P^n\|_{L^2(\underline{y}^n, \infty)}^2 + \Delta \tau \left[\frac{1}{2} \sigma^2 \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 + r \|P^n\|_{L^2(\underline{y}^n, \infty)}^2 + \frac{1}{2} \left(r + \frac{1}{2} \sigma^2\right) (1 - e^{\underline{y}^n})^2 \right] \\ &\leq \|P^n\|_{L^2(\underline{y}^n, \infty)} \|P^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \frac{1}{2} \sigma^2 \Delta \tau e^{\underline{y}^n} (1 - e^{\underline{y}^n}). \end{aligned} \quad (3.2)$$

As we know, $\underline{y}^n \in (-\log(1 + \sigma^2/2r), 0)$ from Carr and Faguet [8], then we obtain the estimate

$$\|P^n\|_{L^2(\underline{y}^n, \infty)}^2 \leq \|P^n\|_{L^2(\underline{y}^n, \infty)} \|P^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \Delta \tau C(r, \sigma). \quad (3.3)$$

The constant $C(r, \sigma)$ on the right-hand side is independent of n , and we write it as C in the following context. Therefore,

$$\begin{aligned} \|P^n\|_{L^2(\underline{y}^n, \infty)} &\leq \|P^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \Delta \tau C \leq \|P^0\|_{L^2(\underline{y}^n, \infty)} + n \Delta \tau C \\ &\leq \|(1 - e^y)^+\|_{L^2(\underline{y}^n, \infty)} + CT \leq C(1 + T). \end{aligned}$$

Taking account of the definition (2.9), we then have the estimate

$$\|P^n\|_{L^2(\underline{y}(\infty), \infty)} \leq C. \quad (3.4)$$

Next, we obtain the estimate for $\|P_y^n\|_{L^2(\underline{y}(\infty), \infty)}$. Multiplying the two sides of (2.6) by $-P_{yy}^n(y)$, and then integrating on the interval $(\underline{y}^n, \infty)$, we have

$$-\int_{\underline{y}^n}^{\infty} [P^n(y) - \Delta \tau \mathcal{L} P^n(y)] P_{yy}^n(y) dy = -\int_{\underline{y}^n}^{\infty} P^{n-1}(y) P_{yy}^n(y) dy. \quad (3.5)$$

We consider the left-hand side of the relation (3.5),

$$\begin{aligned} \int_{\underline{y}^n}^{\infty} \mathcal{L} P^n(y) P_{yy}^n(y) dy &= \frac{1}{2} \sigma^2 \|P_{yy}^n\|_{L^2(\underline{y}^n, \infty)}^2 + r \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 \\ &\quad - r P^n(y) P_y^n(y)|_{\underline{y}^n}^{\infty} + \frac{1}{2} \left(r - \frac{1}{2} \sigma^2\right) [P_y^n(y)]^2|_{\underline{y}^n}^{\infty}, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} -\int_{\underline{y}^n}^{\infty} P^n(y) P_{yy}^n(y) dy &= \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 - P^n(y) P_y^n(y)|_{\underline{y}^n}^{\infty} \\ &= \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 - e^{\underline{y}^n} (1 - e^{\underline{y}^n}). \end{aligned} \quad (3.7)$$

Then we consider the right-hand side of the relation (3.5),

$$-\int_{\underline{y}^n}^{\infty} P^{n-1}(y) P_{yy}^n(y) dy \leq \|P_y^{n-1}\|_{L^2(\underline{y}^n, \infty)} \|P_y^n\|_{L^2(\underline{y}^n, \infty)} - e^{\underline{y}^n} (1 - e^{\underline{y}^n}). \quad (3.8)$$

Summing up (3.5)–(3.8), we then have

$$\begin{aligned} & \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 + \Delta\tau \left[\frac{1}{2} \sigma^2 \|P_{yy}^n\|_{L^2(\underline{y}^n, \infty)}^2 + r \|P_y^n\|_{L^2(\underline{y}^n, \infty)}^2 \right] \\ & \leq \|P_y^n\|_{L^2(\underline{y}^n, \infty)} \|P_y^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \Delta\tau \left[\frac{1}{2} \left(r - \frac{1}{2} \sigma^2 \right) e^{2\underline{y}^n} + r e^{\underline{y}^n} (1 - e^{\underline{y}^n}) \right] \\ & \leq \|P_y^n\|_{L^2(\underline{y}^n, \infty)} \|P_y^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \Delta\tau C. \end{aligned} \quad (3.9)$$

Therefore,

$$\begin{aligned} \|P_y^n\|_{L^2(\underline{y}^n, \infty)} & \leq \|P_y^{n-1}\|_{L^2(\underline{y}^n, \infty)} + \Delta\tau C \leq \|P_y^0\|_{L^2(\underline{y}^n, \infty)} + n\Delta\tau C \\ & \leq \|(1 - e^y)_y^+\|_{L^2(\underline{y}^n, \infty)} + CT \leq C(1 + T). \end{aligned}$$

Taking (2.9) into account, we then have

$$\|P_y^n\|_{L^2(\underline{y}(\infty), \infty)} \leq C. \quad (3.10)$$

Summing up (3.4) and (3.10), we then have

Theorem 1. For the approximate solution $P^n(y)$ to problem (2.6)–(2.7), the estimation

$$\|P^n\|_{H^1(\underline{y}(\infty), \infty)} \leq C$$

is valid.

4. Convergence

First, we note a simple fact: for any $\tau \in (0, T)$, there exists a sequence $\{n\Delta\tau\}$, such that $\lim_{n \rightarrow \infty} n\Delta\tau = \tau$ for fixed T , where $\Delta\tau = \frac{T}{N+1}$ and $n = 0, 1, \dots, N$.

Theorem 1 shows that the sequence $\{P^n(y)\}$ in the function space $H^1(\underline{y}(\infty), \infty)$ is uniformly bounded on n . From the theory of functional analysis (Adams [1]), for any $\tau \in (0, T)$, there exists a subsequence (we also denote as $\{P^n(y)\}$) and a function $\tilde{P}(y, \tau) \in L^\infty((0, T); H^1(\underline{y}(\infty), \infty))$, such that, the sequence $\{P^n(y)\}$ weakly converges to $\tilde{P}(y, \tau)$ in $H^1(\underline{y}(\infty), \infty)$, that is,

$$\lim_{n \rightarrow \infty} \int_{\underline{y}(\infty)}^{\infty} P^n(y) \phi(y) dy = \int_{\underline{y}(\infty)}^{\infty} \tilde{P}(y, \tau) \phi(y) dy \quad \text{for all } \phi \in H^1(\underline{y}(\infty), \infty). \quad (4.1)$$

According to the compactness of embedding theorem of Sobolev space (Adams [1]), we can choose some subsequence of $\{P^n(y)\}$ (which we also denote as $\{P^n(y)\}$) again such that $\{P^n(y)\}$ strongly converges to $\tilde{P}(y, \tau)$ in $C(\underline{y}(\infty), \infty)$, that is,

$$\lim_{n \rightarrow \infty} \|P^n(y) - \tilde{P}(y, \tau)\|_{C(\underline{y}(\infty), \infty)} = 0. \quad (4.2)$$

In particular, on the boundary $y = \underline{y}^n \in (\underline{y}(\infty), \infty)$, we have

$$\lim_{n \rightarrow \infty} P^n(\underline{y}^n) = \tilde{P}(\tilde{y}(\tau), \tau),$$

and then,

$$\lim_{n \rightarrow \infty} \underline{y}^n = \tilde{y}(\tau),$$

where $\tilde{y}(\tau)$ is a function of τ in $(\underline{y}(\infty), \infty)$. In fact,

$$\lim_{n \rightarrow \infty} P^n(\underline{y}^n) = \lim_{n \rightarrow \infty} 1 - e^{\underline{y}^n} = 1 - e^{\tilde{y}(\tau)} = \tilde{P}(\tilde{y}(\tau), \tau).$$

We next prove that the limit function $\tilde{P}(y, \tau)$ is the generalized solution $P(y, \tau)$ to the problem (2.4)–(2.5). It is well known that, for the solution $P(y, \tau)$ to the problem (2.4)–(2.5), for any $y \in (\underline{y}(\tau), \infty)$ and sufficiently small $\Delta\tau$, there is a positive real number $\varepsilon > 0$, such that

$$\left| \frac{P(y, n\Delta\tau) - P(y, (n-1)\Delta\tau)}{\Delta\tau} - P_\tau(y, n\Delta\tau) \right| < \varepsilon,$$

then

$$\left| \frac{P(y, n\Delta\tau) - P(y, (n-1)\Delta\tau)}{\Delta\tau} - \mathcal{L}P(y, n\Delta\tau) \right| < \varepsilon. \quad (4.3)$$

We assume $\tilde{y}(\tau) \leq \underline{y}(\tau)$. Then, for sufficiently large n , $\underline{y}^n \leq \underline{y}(\tau)$ (our result can be also proved in the setting $\underline{y}(\tau) \leq \tilde{y}(\tau)$). Let $Q^n(y) = P^n(y) - P(y, n\Delta\tau)$. Due to the problem (2.6)–(2.7), the definition (2.9) and the relation (4.3), for any $y \in (\underline{y}(\tau), \infty)$ and sufficiently small $\Delta\tau$, we see

$$\frac{Q^n(y) - Q^{n-1}(y)}{\Delta\tau} = \mathcal{L}Q^n(y) + o(1). \quad (4.4)$$

Note the following fact:

$$Q^0(y) = P^0(y) - P(y, 0) = 0 \quad (4.5)$$

and

$$Q^n(y) = 0, \quad y \in [\underline{y}(\infty), \underline{y}(\tau)].$$

We can form the estimate for $\|Q^n\|_{L^2(\underline{y}(\infty), \infty)}$ and $\|Q_y^n\|_{L^2(\underline{y}(\infty), \infty)}$ by using the method of Section 3. Multiplying the two sides of the relation (4.4) by $Q^n(y)$, and integrating on the interval $(\underline{y}^n, \infty)$, we get

$$\|Q^n\|_{L^2(\underline{y}(\infty), \infty)} = o(1). \quad (4.6)$$

Multiplying the two sides of the relation (4.4) by $Q_{yy}^n(y)$, and integrating on the interval $(\underline{y}^n, \infty)$, we have

$$\|Q_y^n\|_{L^2(\underline{y}(\infty), \infty)} = o(1). \quad (4.7)$$

Clearly, the relations (4.6) and (4.7) imply

$$\lim_{n \rightarrow \infty} \|Q^n\|_{H^1(\underline{y}(\infty), \infty)} = 0, \quad (4.8)$$

that is,

$$\lim_{n \rightarrow \infty} P^n(y) = \lim_{n \rightarrow \infty} P(y, n\Delta\tau) = P(y, \tau) \quad \text{in } H^1(\underline{y}(\infty), \infty).$$

According to the compactness of embedding theorem in Sobolev space, we then have

$$\begin{cases} \tilde{P}(y, \tau) = P(y, \tau) & \text{for all } (y, \tau) \in (\underline{y}(\infty), \infty) \times (0, T), \\ \tilde{P}_y(y, \tau) = P_y(y, \tau) & \text{almost everywhere in } (\underline{y}(\infty), \infty) \times (0, T). \end{cases} \quad (4.9)$$

Finally, we prove that $\frac{P^n(y) - P^{n-1}(y)}{\Delta\tau}$ weakly converges to $P_\tau(y, \tau)$. That is, for any $\phi(y) \in H_0^1(\underline{y}(\tau), \infty)$,

$$\lim_{n \rightarrow \infty} \int_{\underline{y}(\tau)}^{\infty} \frac{P^n(y) - P^{n-1}(y)}{\Delta\tau} \phi(y) dy = \int_{\underline{y}(\tau)}^{\infty} P_\tau(y, \tau) \phi(y) dy.$$

In fact, for any $\phi(y) \in H_0^1(\underline{y}(\tau), \infty)$ and sufficiently large n , we have

$$\begin{aligned} & \int_{\underline{y}(\tau)}^{\infty} \frac{Q^n(y) - Q^{n-1}(y)}{\Delta\tau} \phi(y) dy \\ &= \int_{\underline{y}(\tau)}^{\infty} \mathcal{L}Q^n(y) \phi(y) dy + o(1) \int_{\underline{y}(\tau)}^{\infty} \phi(y) dy \\ &= \int_{\underline{y}(\tau)}^{\infty} \left[-\frac{1}{2} \sigma^2 Q_y^n(y) \phi_y(y) + \left(r - \frac{1}{2} \sigma^2 \right) Q_y^n(y) \phi(y) - r Q^n(y) \phi(y) \right] dy + o(1). \end{aligned} \quad (4.10)$$

The first term of the right-hand side of the relation (4.10) is $o(1)$ from the relation (4.8), thus

$$\lim_{n \rightarrow \infty} \int_{\underline{y}(\tau)}^{\infty} \frac{Q^n(y) - Q^{n-1}(y)}{\Delta\tau} \phi(y) dy = 0,$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\underline{y}(\tau)}^{\infty} \frac{P^n(y) - P^{n-1}(y)}{\Delta\tau} \phi(y) dy &= \lim_{n \rightarrow \infty} \int_{\underline{y}(\tau)}^{\infty} \frac{P(y, n\Delta\tau) - P(y, (n-1)\Delta\tau)}{\Delta\tau} \phi(y) dy \\ &= \int_{\underline{y}(\tau)}^{\infty} P_\tau(y, \tau) \phi(y) dy. \end{aligned} \quad (4.11)$$

The relation (4.11) implies that, for any $\tau \in (0, T)$, $\frac{P^n(y) - P^{n-1}(y)}{\Delta\tau}$ weakly converges to $P_\tau(y, \tau)$ in $L_2(\underline{y}(\tau), \infty)$. Summing up the relations (4.9) and (4.11), we then have:

Theorem 2. *The limit function $\tilde{P}(y, \tau)$ of the solution sequence $\{P^n(y)\}$ of problem (2.6) is the generalized solution $P(y, \tau)$ of problem (2.4), that is, $\tilde{P}(y, \tau)$ satisfies integral equation (B.2), and $\lim_{n \rightarrow \infty} \underline{y}^n = \underline{y}(\tau)$.*

5. Concluding remarks

It has been widely acknowledged that there cannot be a closed form solution for the American option pricing problem in a Black–Scholes analysis. A lot of effort have been made to approximate the solution using numerical, semi-analytic and analytic approaches. Though empirical and simulation results have been reported to yield favorable performance for some, no rigorous convergence proof has been established for all of them (Barone-Adesi and Whaley [3], Broadie [6], Geske and Johnson [10], Grant et al. [11], Huang et al. [12], MacMillan [16], Parkinson [20], and Tilly [22]). In this work, we apply techniques developed in functional analysis to prove the convergence of the analytic method of lines by Carr and Faguet [8] (or equivalently, the randomization method of Carr [7]), a semi-analytic method observed to be efficient and accurate in empirical testing.

Acknowledgments

We thank John Randal for his help in the revision process. We are indebted to the anonymous referees for their suggestions and comments. Of course, we are responsible for any remaining shortcomings.

Appendix A

This appendix collects the results of analytical method of lines (Carr and Faguet [8]) for the American put option price $P^n(S)$, and the exercise boundary \underline{S}^n , at an arbitrary time layer $\tau = n\Delta\tau$, $n = 1, \dots, N$, $\Delta\tau = \frac{T}{N+1}$. The general formula of the analytical method of lines for the American put option will be given by solving an ordinary differential equation with the free boundary:

$$\begin{cases} P^n(S) - \Delta\tau \mathcal{L} P^n(S) = P^{n-1}(S), & S \in (\underline{S}^n, \infty), \\ P^n(\underline{S}^n) = 1 - \underline{S}^n, \\ P_S^n(\underline{S}^n) = -1, \\ P^n(\infty) = 0, \end{cases} \quad (\text{A.1})$$

where the operator $\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r$.

Let²

$$\begin{cases} R = \frac{1}{1+r\Delta\tau}, & \alpha = \frac{1}{2} - \frac{r}{\sigma^2} + \frac{\rho}{2}, & \beta = \frac{1}{2} - \frac{r}{\sigma^2} - \frac{\rho}{2}, \\ \rho = 2\sqrt{\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2 + \frac{1}{\sigma^2\Delta\tau}}, \\ \theta_j(S, \rho) = \sum_{i=1}^j \frac{(-1)^{i+j}}{i! \rho^{2j-i}} \left[\binom{2j-2-i}{j-2} - \binom{2j-2-i}{j} \right] \log^i S, & j = 1, \dots, n-1. \end{cases} \quad (\text{A.2})$$

Then we have

$$P^n(S) = \begin{cases} \pi_0^n(S) + A_0^n S^\alpha + B_0^n S^\beta, & 1 \leq S \leq \infty, \\ \pi_1^n(S) + A_1^n S^\alpha + B_1^n S^\beta, & \underline{S}^1 \leq S \leq 1, \\ \dots \\ \pi_n^n(S) + A_n^n S^\alpha + B_n^n S^\beta, & \underline{S}^n \leq S \leq \underline{S}^{n-1}, \end{cases}$$

² To implement (A.2), note $\binom{-1}{-1} = 0$, $\binom{-1}{1} = -1$, $\binom{0}{0} = 1$, $\binom{1}{2} = 0$, $\binom{0}{2} = 0$, $\binom{0}{0} = 0$. See Carr and Faguet [8].

and

$$\underline{S}^n = \left[\frac{\beta(R-1)}{\rho A_n^n} \right]^{\frac{1}{\alpha}}.$$

Here

$$\begin{aligned} \pi_0^n(S) &= \sum_{j=1}^{n-1} \left[\frac{-2}{\sigma^2 \Delta \tau} \right]^j B_0^{n-j} S^\beta \theta_i(S, -\rho), \\ \pi_i^n(S) &= R^{n-i+1} + \sum_{j=1}^{n-1} \left[\frac{-2}{\sigma^2 \Delta \tau} \right]^j [A_i^{n-j} S^\alpha \theta_j(S, \rho) + B_i^{n-j} S^\beta \theta_i(S, -\rho)], \\ i &= 1, \dots, n+1, \end{aligned}$$

and

$$A_0^n = 0, \quad A_i^n = A_{i-1}^n + \frac{\beta \Delta \pi_i^n(\underline{S}^{i-1}) - \underline{S}^{i-1} [\Delta \pi_i^n]'(\underline{S}^{i-1})}{\rho (\underline{S}^{i-1})^\alpha},$$

where $\Delta \pi_i^n(S) = \pi_i^n(S) - \pi_{i-1}^n(S)$, $i = 1, \dots, n$.

In particular, at the first time layer $\tau = \Delta \tau$, the problem is to solve the following ordinary differential equation

$$P^1(S) - \Delta \tau \mathcal{L} P^1(S) = (1-S)^+, \quad S \in (\underline{S}^1, \infty),$$

with the boundary conditions

$$\begin{cases} P^1(\underline{S}^1) = 1 - \underline{S}^1, \\ P_S^1(\underline{S}^1) = -1, \\ P^1(\infty) = 0. \end{cases}$$

The solutions $P^1(S)$ and \underline{S}^1 are

$$P^1(S) = \begin{cases} \pi_0^1(S) + A_0^1 S^\alpha + B_0^1 S^\beta, & 1 \leq S \leq \infty, \\ \pi_1^1(S) + A_1^1 S^\alpha + B_1^1 S^\beta, & \underline{S}^1 \leq S \leq 1, \end{cases} \quad (\text{A.3})$$

and

$$\underline{S}^1 = \left[\frac{\beta(R-1)}{\rho A_1^1} \right]^{\frac{1}{\alpha}},$$

where $\pi_0^1(S) = 0$, $\pi_1^1(S) = R$ and $\pi_2^1(S) = 1 - S$, with R , α and β given by (A.2) and $A_0^1 = 0$, $A_1^1 = \frac{\beta R}{\rho}$.

Appendix B

In this appendix, we will introduce some concepts about function space, strong convergence, weak convergence and generalized solution to the problem (1.2)–(1.3), etc.

Suppose $f(x)$ is a real function defined on the interval $(a, b) \subseteq (-\infty, \infty)$. Let us provide some definitions of function spaces needed in this paper as

$$L^2(a, b) = \left\{ f(x) \left| \int_a^b f^2(x) dx < \infty \right. \right\},$$

$$H^l(a, b) = \left\{ f(x) \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} \in L^2(a, b), \alpha = 0, \dots, l \right. \right\},$$

where l is a positive integer, the notation $\frac{\partial^\alpha f(x)}{\partial x^\alpha}$ is the generalized derivative of $f(x)$, i.e., for any $\phi(x) \in C_0^\infty(a, b)$,

$$\int_a^b \frac{\partial^\alpha}{\partial x^\alpha} f(x) \phi(x) dx = (-1)^\alpha \int_a^b f(x) \frac{\partial^\alpha \phi(x)}{\partial x^\alpha} dx$$

and

$$H^0(a, b) = L^2(a, b),$$

$$H_0^l(a, b) = \{ f(x) \in H^l(a, b) \mid \text{support } f(x) \subseteq (a, b) \}.$$

The function space $H^l(a, b)$ or $H_0^l(a, b)$ is the so-called Sobolev space, and it is also a Hilbert space, so we can introduce the inner product in the space $H^l(a, b)$ or $H_0^l(a, b)$ so that, for any f and $g \in H^l(a, b)$ or $H_0^l(a, b)$,

$$(f, g)_{(a, b), l} \equiv \sum_{\alpha=0}^l (f, g)_{(a, b), \alpha} \equiv \sum_{\alpha=0}^l \int_a^b \frac{\partial^\alpha f(x)}{\partial x^\alpha} \frac{\partial^\alpha g(x)}{\partial x^\alpha} dx$$

and the norm in the space $H^l(a, b)$ or $H_0^l(a, b)$ is

$$\|f\|_{H^l(a, b)} \equiv (f, f)_{(a, b), l}^{\frac{1}{2}} \equiv \left[\sum_{\alpha=0}^l \int_a^b \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} \right|^2 dx \right]^{\frac{1}{2}}.$$

Finally, we give three important definitions from Adams [1] and Loins [15] as follows.

Definition B.1 (*Strong convergence*). Suppose $\{f^n(x)\}$ is a function sequence in the space $H^l(a, b)$ or $H_0^l(a, b)$, and a function $f(x) \in H^l(a, b)$ or $H_0^l(a, b)$, we say $f^n(x)$ strongly converges to $f(x)$ means that

$$\lim_{n \rightarrow \infty} \|f^n(x) - f(x)\|_{H^l(a, b)} = 0.$$

Definition B.2 (*Weak convergence*). Suppose $\{f^n(x)\}$ is a function sequence in the space $H^l(a, b)$ or $H_0^l(a, b)$, and a function $f(x) \in H^l(a, b)$ or $H_0^l(a, b)$, we say $f^n(x)$ weakly converges to $f(x)$ means that, for any $\phi(x) \in H^l(a, b)$ or $H_0^l(a, b)$,

$$\lim_{n \rightarrow \infty} \int_a^b f^n(x) \phi(x) dx = \int_a^b f(x) \phi(x) dx. \quad (\text{B.1})$$

Definition B.3 (*Generalized solution*). Suppose the function $P(y, \tau) \in H^1(\underline{(y(\tau), \infty)} \times (0, T))$, for any $\tau \in (0, T)$, satisfies, for any $\phi(y) \in H_0^1(\underline{(y(\tau), \infty)})$,

$$\begin{aligned}
& \int_{\underline{y}(\tau)}^{\infty} P_{\tau}(y, \tau) \phi(y) dy \\
&= \int_{\underline{y}(\tau)}^{\infty} \left[-\frac{1}{2} \sigma^2 P_y(y, \tau) \phi_y(y) + \left(r - \frac{1}{2} \sigma^2 \right) P_y(y, \tau) \phi(y) - r P(y, \tau) \phi(y) \right] dy \quad (\text{B.2})
\end{aligned}$$

then we say $P(y, \tau)$ is a generalized solution to the problem (1.2)–(1.3). Here the function space

$$\begin{aligned}
& H^1(\underline{y}(\tau), \infty) \times (0, T) \\
&= \{f(y, \tau) \mid f(y, \tau), f_y(y, \tau), f_{\tau}(y, \tau) \in L^2(\underline{y}(\tau), \infty) \times (0, T)\}.
\end{aligned}$$

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